Journal of Statistics: Advances in Theory and Applications
Volume 1, Number 1, 2009, Pages 49-63

TESTS OF FIT FOR ASYMMETRIC LAPLACE DISTRIBUTIONS WITH APPLICATIONS*

KOSTAS FRAGIADAKIS and SIMOS G. MEINTANIS

Department of Economics
National and Kapodistrian University of Athens
8 Pesmazoglou Street
105 59 Athens
Greece

Abstract

Consistent goodness-of-fit tests for the family of asymmetric Laplace distributions are constructed. The tests are based on a weighted integral incorporating the empirical characteristic function of the data suitably standardized via the maximum likelihood estimators. Finite-sample comparison with classical procedures is provided, as well as applications with financial data.

1. Introduction

The purpose of this paper is to provide goodness-of-fit tests for a generalization of the classical Laplace distribution, the so-called asymmetric Laplace distribution (ALD). Recall that the density function of the Laplace distribution (also known as the first law of Laplace) is

$$f_{\delta, \sigma}(x) = \frac{1}{2\sigma} e^{-\frac{|x-\delta|}{\sigma}}, \; -\infty < x < \infty, \tag{1.1}$$

2000 Mathematics Subject Classification: 60E10.

Keywords and phrases: Laplace distribution, goodness-of-fit test, characteristic function, maximum likelihood estimation, skew Laplace distribution, quantitative finance.

*A first draft of this paper was presented at the International Electronic Conference on Computer Science 2007.

Received March 28, 2009

where $-\infty < \delta < \infty$ and $\sigma > 0$.

The ALD results from this density function by introducing to the already existing location and scale parameters δ and σ, respectively, an extra (shape) parameter k. Then, the density of the ALD becomes

$$f_{\delta, \sigma, k}(x) = \frac{1}{\sigma} \cdot \frac{k}{1 + k^2} \begin{cases} \exp\left(-\frac{k}{\sigma}(x - \delta)\right) & \text{for } x \geq \delta \\ \exp\left(\frac{1}{\sigma k}(x - \delta)\right) & \text{for } x < \delta, \end{cases} \tag{1.2}$$

where $k > 0$. In fact there are various forms of skewed Laplace distributions but here we will follow the definition of Kotz et al. [8], and write $AL(\delta, \sigma, k)$ for the three- parameter family with density given by Equation (1.2). Note that k regulates the shape of the underlying distribution. In particular, setting $k = 1$ in (1.2) yields the classical Laplace distribution with density function given by (1.1), which is the only symmetric member of the ALD class. On the other hand if $k \neq 1$, the corresponding distribution is asymmetric with excess kurtosis.

Suppose that on the basis of independent copies $X_1, X_2, \ldots, X_n$, of a random variable X, we wish to test the null hypothesis

H_0 : The law of X is $AL(\delta, \sigma, k)$ for some $\delta \in \mathbb{R}, \sigma > 0$ and $k > 0$.

Motivation for considering the ALD stems from the fact that this distribution has tails heavier than the normal distribution, and by incorporating extra skewness, becomes an excellent choice of model for applications in Economics, and particularly for modeling financial data. See for instance, Kotz et al. [8], Kozubowski and Podgórski [10], and references therein.

The rest of the paper is organized as follows. Section 2 presents the new test statistic, while Section 3 is devoted to estimation of parameters. In Section 4 the consistency of the test is proved under certain conditions on the behavior of the estimators under alternative distributions. In Section 4 Monte Carlo simulations are presented examining the power of the new tests in comparison to more classical procedures. Finally in Section 5 applications in real financial data are provided.

2. Test Statistics

If $X \sim AL(\delta, \sigma, k)$, then the characteristic function (CF), $\phi(t) = \mathbf{E}(e^{itX})$ of X, is given by

$$\phi(t) = \frac{e^{i\delta t}}{1 + \sigma^2 t^2 - i(\frac{1}{k} - k)\sigma t}, \tag{2.1}$$

and satisfies

$$D(t;\ \vartheta) = 0,\ t \in \mathbb{R}, \tag{2.2}$$

where

$$D(t;\ \vartheta) = (1 + \sigma^2 t^2 - i\vartheta t)\phi(t) - \exp(i\delta t)$$

with $\vartheta = (1/k) - k$. In view of (2.2), it is natural to construct a test statistic based on a measure of deviation from zero of the function

$$D_n(t;\ \hat{\vartheta}_n) = (1 + t^2 - i\hat{\vartheta}_n t)\phi_n(t) - 1,$$

where $\hat{\vartheta}_n = (1/\hat{k}_n) - \hat{k}_n$,

$$\phi_n(t) = \frac{1}{n}\sum_{j=1}^{n} e^{itY_j},$$

is the empirical CF of the standardized data $Y_j = (X_j - \hat{\delta}_n)/\hat{\sigma}_n$, $j = 1, 2, \ldots, n$, and $(\hat{\delta}_n, \hat{\sigma}_n, \hat{k}_n)$ denote the maximum likelihood estimator of (δ, σ, k). Specifically we suggest to reject the null hypothesis H_0 for large values of

$$\hat{T}_{n,w} = n\int_{-\infty}^{\infty} |D_n(t;\ \hat{\vartheta}_n)|^2 w(t)dt, \tag{2.3}$$

with $w(t)$ denoting a non-negative weight function. From (2.3) we have by straightforward algebra

$$\hat{T}_{n,w} = \frac{1}{n}\sum_{j,k=1}^{n}\left[I_c^{(4)}(Y_j - Y_k) + (2 + \hat{\vartheta}_n^2)I_c^{(2)}(Y_j - Y_k) + I_c^{(0)}(Y_j - Y_k)\right]$$

$$+ nI_c^{(0)}(0) - 2\sum_{j=1}^{n}\left[I_c^{(2)}(Y_j) + \hat{\vartheta}_n I_s(Y_j) + I_c^{(0)}(Y_j)\right], \tag{2.4}$$

where

$$I_c^{(m)}(b) = \int_{-\infty}^{+\infty} t^m \cos(bt)w(t)dt, \quad m = 0,\ 2,\ 4,$$

and

$$I_s(b) = \int_{-\infty}^{+\infty} t\sin(bt)w(t)dt.$$

Although theoretical properties of the test statistic remain qualitatively invariant, provided that $w(t)$ satisfies some general conditions, particular appeal lies with weight functions that render the test statistic in a closed formula suitable for computer implementation. For instance if one chooses the function $w(t) = e^{-a|t|}$, $a > 0$, all integrals figuring in (2.4) can easily be computed. In particular, we have

$$I_c^{(0)}(b) = \int_{-\infty}^{+\infty} \cos(bt)e^{-a|t|}dt = \frac{2a}{a^2 + b^2},$$

$$I_c^{(2)}(b) = \int_{-\infty}^{+\infty} t^2\cos(bt)e^{-a|t|}dt = -4a\,\frac{3b^2 - a^2}{(a^2 + b^2)^3},$$

$$I_c^{(4)}(b) = \int_{-\infty}^{+\infty} t^4\cos(bt)e^{-a|t|}dt = 48a\,\frac{5b^4 + a^4 - 10a^2b^2}{(a^2 + b^2)^5},$$

and

$$I_s(b) = \int_{-\infty}^{+\infty} t \sin(bt) e^{-a|t|} dt = \frac{4ab}{(a^2 + b^2)^2}.$$

In closing this section we note that the empirical CF has recently proved to be a powerful tool for statistical inference. In goodness-of-fit problems particularly, methods based on the empirical CF are not only convenient but also very competitive to more classical procedures such as those based on the empirical distribution function. The reader is referred to Koutrouvelis and Meintanis [9], Gürtler and Henze [4], Meintanis [12, 13], Klar and Meintanis [6], and Epps [2], to name just a few of the more recent works. A large part of the earlier literature on the empirical CF may be found in Ushakov [14].

3. Estimation of Parameters

As noted in the previous section, the new test statistic incorporates the maximum likelihood (ML) estimators of the parameters $-\infty < \delta < \infty$, $\sigma > 0$, and $k > 0$. These estimators were proposed and fully studied by Kotz et al. [7]. The computational part is presented here for completeness. For further theoretical properties of the ML estimators of the ALD the reader is referred to Kotz et al. [7]. Let $x_1, x_2, \ldots, x_n$, denote particular realizations of $X_1, X_2, \ldots, X_n$, and write $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$, for the corresponding realizations of the order statistics $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$. Then from (1.2), it follows that the average (divided by n) log-likelihood function is given by

$$Q(\delta, \sigma, k) = \ln \frac{k}{1 + k^2} - \ln \sigma - \frac{1}{\sigma}\left(\frac{k}{n}\sum_{j=1}^{n}(x_j - \delta)^+ + \frac{1}{kn}\sum_{j=1}^{n}(x_j - \delta)^-\right) \quad (3.1)$$

where,

$$(x_j - \delta)^+ = \begin{cases} x_j - \delta, & x_j \geq \delta \\ 0, & x_j < \delta, \end{cases} \quad \text{and} \quad (x_j - \delta)^- = \begin{cases} \delta - x_j, & x_j \leq \delta \\ 0, & x_j > \delta. \end{cases}$$

To obtain the ML estimators we follow the steps below:

Step 1

For $\delta = x_j,\ j = 1, 2, \ldots, n,$ compute

$$H(\delta) = 2\ln\left(\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \delta)^+} + \sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \delta)^-}\right)$$

Step 2

Set $\hat{\delta}_n = x_{(r)}$, where $x_{(r)}$ is such that: $H(x_{(r)}) \le H(x_{(j)})$ for all $j = 1, 2, \ldots, n.$

Step 3

Case I: If $r = 1$ or $r = n \Rightarrow$ the ML estimators do not exist.

Case II: If $1 < r < n,$ then set $\hat{\sigma}_n = \sigma(\hat{\delta}_n),$ and $\hat{k}_n = k(\hat{\delta}_n),$ where

$$\sigma(\delta) = \left(4\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \delta)^-}\right)\left(4\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \delta)^+}\right)\left(4\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \delta)^-} + 4\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \delta)^+}\right),$$

and

$$k(\delta) = 4\sqrt{\frac{n^{-1}\sum_{j=1}^{n}(x_j - \delta)^-}{n^{-1}\sum_{j=1}^{n}(x_j - \delta)^+}}.$$

Note, that this algorithm is a modification of the algorithm proposed by Kotz et al. [7]. We notice in particular, that proper modification of the function $H(\delta)$ provides more accurate and efficient estimators.

4. Consistency of the Test

Under certain assumptions, the test statistic figuring in (2.3) is consistent against general alternatives to the asymmetric Laplace law. To see this assume that under the (otherwise arbitrary) law for the random variable X, we have that $(\hat{\delta}_n, \hat{\sigma}_n, \hat{k}_n) \to (\tilde{\delta}, \tilde{\sigma}, \tilde{k})$ holds almost surely as

$n \to \infty$, for some $\tilde{\delta} \in \mathbb{R}$, and $\tilde{\sigma}, \tilde{k} > 0$. Also denote by $\tilde{\varphi}(t)$ the CF of the random variable $(X - \tilde{\delta}) / \tilde{\sigma}$. Then the uniform strong consistency of the empirical CF (refer to Feuerverger and Mureika [3], Csörgő, [1], implies that if $t_n \to t$, then

$$|\phi_n(t_n) - \tilde{\varphi}(t)| \to 0,$$

almost surely, as $n \to \infty$. Hence one has $D_n(t; \hat{\vartheta}_n) \to (1 + t^2 - i\tilde{\vartheta}t)\tilde{\varphi}(t) - 1 := \tilde{D}(t; \tilde{\vartheta})$, where $\tilde{\vartheta} = (1/\tilde{k}) - \tilde{k}$, and consequently that,

$$|D_n(t; \hat{\vartheta}_n)|^2 \to |\tilde{D}(t; \tilde{\vartheta})|^2.$$

In turn representation (2.3) implies that

$$\liminf_{n\to\infty} \frac{\hat{T}_{n,w}}{n} \geq \Delta_w := \int_{-\infty}^{+\infty} |\tilde{D}(t; \tilde{\vartheta})|^2 w(t)dt,$$

by Fatou's Lemma. Assuming that $w(t)$ does not vanish (apart from a set of measure zero), we have that Δ_w is positive, and consequently that $\hat{T}_{n,w} \to \infty$, almost surely, as $n \to \infty$, unless $\tilde{D}(t; \tilde{\vartheta}) = 0$, $t \in \mathbb{R}$. But $\tilde{D} \equiv 0$, implies that $\tilde{\varphi}(t)$ is identically equal to $\phi_0(t) = (1 + t^2 - i\tilde{\vartheta}t)^{-1}$. Notice however that $\phi_0(t)$ corresponds to the ALD with $(\delta, \sigma) = (0, 1)$, and shape parameter equal to $\tilde{k}$, which implies that the CF of X is given by $\phi(t)$ of Equation (2.1), for some $(\tilde{\delta}, \tilde{\sigma}, \tilde{k})$. In turn, from the uniqueness of the CF it follows that the law of X is $AL(\tilde{\delta}, \tilde{\sigma}, \tilde{k})$, and the proof of consistency is complete.

5. Simulations

In this section we present the results of a Monte Carlo study for the new test given by (2.4) with weight function $w(t) = \exp(-a|t|)$, denoted by $\hat{T}_{n,a}$, in comparison to classical tests based on the empirical distribution function. Specifically we compare $\hat{T}_{n,a}$ with the Kolmogorov-Smirnov (KS), the Cramér-von Mises (CM), and the Anderson- Darling (AD), tests, which are computed as follows: Denote by $F(x;\,\delta,\,ó,\,k)$ the cumulative distribution function of the ALD corresponding to the density (1.2), and let $\hat{F}(x) = F(x;\,\hat{\delta}_n,\,ó_n,\,\hat{k}_n)$. Then the KS statistic is given by

$$\text{KS} = \sqrt{n}\max(KS^{+},\,KS^{-}),$$

where

$$KS^{+} = \max_{j=1,2,\ldots,n}\left(\frac{j}{n} - \hat{F}(X_{(j)})\right),\quad KS^{-} = \max_{j=1,2,\ldots,n}\left(\hat{F}(X_{(j)}) - \frac{j-1}{n}\right).$$

The CM and AD statistics are given by

$$\text{CM} = \frac{1}{12n} + \sum_{j=1}^{n}\left(\hat{F}(X_{(j)}) - \frac{2j-1}{2n}\right)^2$$

and

$$\text{AD} = -n - \sum_{j=1}^{n}\left(\frac{2j-1}{n}\log\hat{F}(X_{(j)}) + \frac{2(n-j)+1}{n}\log(1-\hat{F}(X_{(j)}))\right).$$

The Monte Carlo study was implemented by drawing 1000 samples of size n. However, it is well known that the limit null distribution of the test statistics depends on the value of the shape parameter ϑ; see for instance Meintanis and Swanepoel [11]. Hence, since ϑ is unknown, we resort to a parametric bootstrap procedure in order to obtain the critical point p_α of a size $-\alpha$ test as follows:

- 1. Conditionally on the observed value of $X_j,\ j = 1, 2, \ldots, n$, compute the ML estimates $(\hat{\delta}_n,\,\hat{\sigma}_n,\,\hat{k}_n)$ and, then the observations $\hat{Y}_j = (X_j - \hat{\delta}_n)/\hat{\sigma}_n,\ j = 1, 2, \ldots, n$.

- 2.a. Calculate the value of the test statistic, say $\hat{T}$, based on $\{\hat{Y}_j\}_{j=1}^{n}$ and $\hat{\vartheta}_n = (1/\hat{k}_n) - \hat{k}_n$.

• 2.b.1. Generate a bootstrap sample X_j^*, $j = 1, 2, \ldots, n$, from AL $(0, 1, \hat{k}_n)$.

• 2.b.2. On the basis of X_j^*, $j = 1, 2, \ldots, n$, compute the estimates $(\hat{\delta}_n^*, \hat{\sigma}_n^*, \hat{k}_n^*)$ and, then the observations $\hat{Y}_j^* = (X_n^* - \hat{\delta}_n^*) / \hat{\sigma}_n^*$, $j = 1, 2, \ldots, n$.

• 2.b.3. Calculate the value of the test statistic, say $\hat{T}^*$, based on $\{\hat{Y}_j^*\}_{j=1}^n$ and $\hat{\vartheta}_n^* = (1 / \hat{k}_n^*) - \hat{k}_n^*$.

• 3. Repeat steps 2.b.1 - 2.b.3., and calculate M values of $\hat{T}^*$, say $\hat{T}_j^*$, $j = 1, 2, \ldots, M$.

• 4. Obtain p_α as $\hat{T}^*{}_{(M-\alpha M)}$, where $\hat{T}_{(j)}^*$, $j = 1, 2, \ldots, M$ denotes the ordered $\hat{T}_j^*$ - value.

In fact, and with $M = 100$, we have used the modified critical points $\tilde{p}_\alpha = \hat{T}^*{}_{(M-\alpha M)} + (1 - \alpha)(\hat{T}^*{}_{(M-\alpha M+1)} - \hat{T}^*{}_{(M-\alpha M)})$, which leads to a more accurate empirical level of the test.

In addition to the asymmetric Laplace distribution $AL(0, 1, k)$, simply denoted by $AL(k)$, the following distributions are simulated:

• The skew normal distribution $SN(\lambda) = (\lambda / (1 + \lambda^2)^{1/2})|Z_1| + (1 / (1 + \lambda^2)^{1/2})Z_2$, where Z_1, Z_2 are independent standard normal variates.

• Tukey's g-distribution, denoted by $TU(g)$, where $TU(g) = (e^{gZ_1} - 1) / g$, with Z_1 is as above.

• The skew t-distribution of Kim [5], denoted by $ST(\lambda, \nu)$, where

$$ST(\lambda, \nu) = \frac{\lambda}{\sqrt{1 + \lambda^2}} \cdot \frac{|Z_1|}{\sigma} + \frac{1}{\sqrt{1 + \lambda^2}} \cdot \frac{|Z_2|}{\sigma},$$

with Z_1, Z_2 as above and σ^2 following a Gamma distribution with shape parameter equal to $\nu / 2$ and scale parameter equal to $2 / \nu$.

These distributions provide alternative ways of modelling skewed data. Moreover, they include the normal law as a particular case: The skew-normal for $\lambda = 0$, the skew t-distribution, $\mathcal{ST}(0, \nu)$, as $\nu \to \infty$, and Tukey's g-distribution as $g \to 0$.

The simulation results for $n = 100$ (resp. $n = 200$) are reported in Table 1 (resp. Table 2) in the form of percentage of rejection rounded to the nearest integer. (For simplicity we write $\hat{T}_a$ in the tables for the new test). These percentages suggest that among the classical procedures the AD is the most powerful test for the alternatives considered. Comparison of powers between the AD and $\hat{T}_a$ tests, favors the former for some alternatives while favoring the latter for other alternatives. However it may be seen that the CF-based test with, say $a = 2.2$, is a strong competitor, and in fact outperforms all classical procedures under the majority of sampling situations with both sample sizes, though not by a wide margin.

6. Applications

Our application deals with exchange currency data. The data consist of daily currency spot exchange rates for USA and Japan, covering the period from January 1, 1975 to December 31, 2005. The data were obtained from the official site of the Bank of England at www.bankofengland.co.uk. The variable of interest is the 'relative' daily return $R_t = (r_t - r_{t-1}) / r_{t-1}$, where r_t denotes the raw return and $t - 1$ and t refer to two consecutive days.

There exist 31 samples, each sample corresponding to one year, with sample size $\approx$ 250. It is common knowledge however that there exists temporal dependence in financial variables in general, and exchange rate returns in particular. Therefore we first estimate an ARMA (1,1) model

$\hat{R}_t = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{R}_{t-1} - \hat{\beta}_1 \hat{\varepsilon}_{t-1}$ and, then apply the test of fit to the ALD on the residuals of this ARMA model.

Our results indicate a good rate of acceptance. Specifically, the rate of acceptance at a 5% level of significance in the spot exchange rate for the 31 different samples is 51:6% (or 16 out of 31 samples). Note that if we exclude the last 8 years, i.e., if we only consider the data corresponding to the first 23 years (1975-1997), the rate of acceptance is increased to 69:57%. The cause in this significance difference could be potentially due to a structural break in the currency exchange rates caused by the introduction of the Euro. Figure 1 provides an additional visual verification in the form of histograms and probability-probability $(\mathrm{P} - \mathrm{P})$ plots. Namely we have plotted the USD/YEN exchange rate for the periods 1985-1987. In the first set of graphs one can see the histograms together with the superimposed fitted AL models, while the second set consists of the corresponding P-P plots. In these graphs too, the agreement between the empirical distribution of the residuals and a corresponding ALD is seen to be quite good.

References

[1] S. Csörgő, Limit behaviour of the empirical characteristic function, Ann. Probab. 9 (1981), 130-144.

[2] T. W. Epps, Tests for location-scale families based on the empirical characteristic function, Metrika 62 (2005), 99-114.

[3] A. Feuerverger and R. Mureika, The empirical characteristic function and its applications, Ann. Statist. 5 (1977), 88-97.

[4] N. Gürtler and N. Henze, Goodness-of-fit tests for the Cauchy distribution based on the empirical characteristic function, Ann. Instit. Statist. Math. 52 (2000), 267-286.

[5] H. J. Kim, Binary regression with a class of skew t link models. Commun. Statist. Theor. Meth. 31 (2002), 1863-1886.

[6] B. Klar and S. Meintanis, Tests for normal mixtures based on the empirical characteristic function. Comput. Statist. Dat. Anal. 49 (2005), 227-242.

[7] S. Kotz, T. J. Kozubowski and K. Podgorski, Maximum likelihood estimation of asymmetric Laplace parameters, Ann. Inst. Statist. Math. 54 (2002), 816-826.

[8] S. Kotz, T. J. Kozubowski and K. Podgorski, The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering and Finance, Birkhauser, Boston, (2001).

[9] I. A. Koutrouvelis and S. G. Meintanis, Testing for stability based on the empirical characteristic function with applications to financial data, J. Statist. Comput. Simulation 64 (1999), 275-300.

[10] T. J. Kozubowski and K. Podgorski, Asymmetric Laplace laws and modeling financial data, Math. Comput. Modelling 34 (2001), 1003-1021.

[11] S. G. Meintanis and J. Swanepoel, Bootstrap goodness-of-fit tests with estimated parameters based on empirical transforms, Statist. Probab. Lett. 77 (2007), 1004-1013.

[12] S. G. Meintanis, A class of omnibus tests for the Laplace distribution based on the empirical characteristic function, Commun. Statist. - Theory Meth. 33 (2004a), 925-948.

[13] S. G. Meintanis, Goodness-of-fit tests for the logistic distribution based on empirical transforms, Sankhyā 66, Part 2 (2004b), 306-326.

[14] N. G. Ushakov, Selected Topics in Characteristic Functions, VSP, Utrecht, (1999).

Table 1. Percentage of rejection for 1000 Monte Carlo samples of size $n = 100$ at 5% (left) 10% (right) level of significance

	$\hat{T}_{1.5}$	$\hat{T}_{1.75}$	$\hat{T}_{2.0}$	$\hat{T}_{2.2}$	KS	CM	AD
AL(0.75)	4 10	4 8	4 9	4 9	4 8	5 11	4 9
AL (0.90)	6 10	6 10	5 10	5 10	5 9	5 10	5 11
AL (1.00)	5 10	5 10	5 9	5 9	5 11	5 10	5 9
AL (1.10)	5 9	5 9	5 10	5 10	5 11	5 10	5 11
AL (1.25)	5 10	5 10	5 10	5 9	5 11	5 10	5 10
SN(0.25)	46 60	55 68	61 74	64 77	43 59	58 71	61 73
SN (05.0)	45 60	54 69	60 74	62 76	43 59	56 70	59 73
SN (1.00)	44 60	52 67	59 72	63 74	43 58	55 70	58 72
SN (1.50)	42 57	50 64	56 70	59 72	41 57	54 68	56 70
TU(0.05)	46 62	55 70	60 75	65 78	45 60	59 73	62 74
TU (0.10)	45 61	54 69	60 74	63 77	44 59	57 72	60 73
TU (0.15)	43 59	52 66	55 70	58 73	43 57	54 69	55 71
TU (0.20)	39 54	47 62	51 66	53 67	39 54	49 65	52 67
ST(0.5,2)	25 34	30 41	35 45	38 47	37 46	43 53	51 60
ST (0.5,4)	14 23	14 24	15 25	15 25	13 23	15 25	16 26
ST (0.5,6)	23 34	26 38	28 40	28 41	19 32	25 37	25 36
ST (0.5,8)	28 41	31 45	35 47	37 48	26 39	30 43	31 44
ST (0.5,10)	33 45	37 51	41 54	44 57	30 44	36 53	37 52
ST (1.0,2)	23 32	27 38	31 42	33 45	38 47	44 53	51 60
ST (1.0,4)	13 22	15 23	14 24	14 24	12 22	14 23	16 24
ST (1.0,6)	22 33	25 37	26 39	26 40	20 31	23 35	24 36
ST (1.0,8)	25 38	29 44	32 46	34 47	25 37	29 42	30 43
ST (1.0,10)	29 41	34 46	36 51	39 53	29 42	34 49	36 50
ST (2.0,2)	19 27	22 32	25 36	27 39	41 51	46 56	51 61
ST (2.0,4)	11 18	12 19	11 20	11 21	11 20	12 22	13 23
ST (2.0,6)	18 29	19 31	21 32	20 32	18 28	21 32	23 33
ST (2.0,8)	21 34	24 37	25 39	26 40	23 35	26 38	26 39
ST(2.0,10)	26 38	30 43	32 46	32 46	27 37	32 45	33 46

Table 2. Percetage of rejection for 1000 Monte Carlo samples of size $n = 200$ at 5% (left) 10% (right) level of significance

	$\hat{T}_{1.5}$	$\hat{T}_{1.75}$	$\hat{T}_{2.0}$	$\hat{T}_{2.2}$	KS	CM	AD
AL(0.75)	6 12	5 12	5 11	5 11	5 10	5 10	4 10
AL (0.90)	6 11	6 11	6 11	6 11	6 12	6 11	5 12
AL (1.00)	5 10	5 10	5 10	5 10	5 11	4 9	5 11
AL (1.10)	5 9	5 9	5 9	5 9	6 10	5 10	6 10
AL (1.25)	5 9	5 9	5 9	5 9	6 11	6 11	6 11
SN(0.25)	86 93	92 96	95 97	96 98	76 88	93 97	95 98
SN (0.50)	86 93	91 96	94 98	95 98	77 87	93 97	95 98
SN (1.00)	85 92	91 96	93 97	95 98	77 88	92 97	94 97
SN (1.50)	84 91	90 94	92 96	93 97	75 87	90 96	93 96
TU(0.05)	87 93	92 97	95 99	96 99	79 89	94 98	95 99
TU (0.10)	87 93	91 96	94 98	96 98	78 89	93 97	94 98
TU (0.15)	86 93	91 95	93 97	94 98	75 87	90 95	93 96
TU (0.20)	84 91	89 94	91 95	92 96	72 84	87 94	91 95
ST(0.5,2)	42 52	50 61	56 66	61 70	56 67	65 73	73 79
ST (0.5,4)	29 42	31 44	31 45	30 43	21 34	27 40	28 40
ST (0.5,6)	51 62	54 67	56 68	57 68	39 54	52 65	52 65
ST (0.5,8)	61 71	65 76	68 88	70 80	51 64	64 77	66 77
ST (0.5,10)	66 77	72 82	75 84	77 86	56 70	72 83	74 83
ST (1.0,2)	41 52	49 60	55 65	59 69	61 69	67 74	74 81
ST (1.0,4)	28 39	28 42	29 41	28 41	22 32	26 36	28 38
ST (1.0,6)	49 62	54 65	56 67	56 67	39 54	50 65	51 64
ST (1.0,8)	60 73	66 78	69 79	70 80	49 65	64 76	66 77
ST (1.0,10)	66 77	72 83	76 85	78 86	55 71	70 82	74 83
ST (2.0,2)	39 52	47 58	53 64	58 66	63 73	70 79	77 84
ST (2.0,4)	25 38	26 39	26 39	25 38	20 31	23 37	26 40
ST (2.0,6)	45 57	48 60	49 63	49 62	36 48	43 58	46 61
ST (2.0,8)	54 67	59 72	60 72	60 72	43 57	55 69	58 71
ST(2.0,10)	61 75	67 78	69 79	69 80	50 65	65 77	68 80

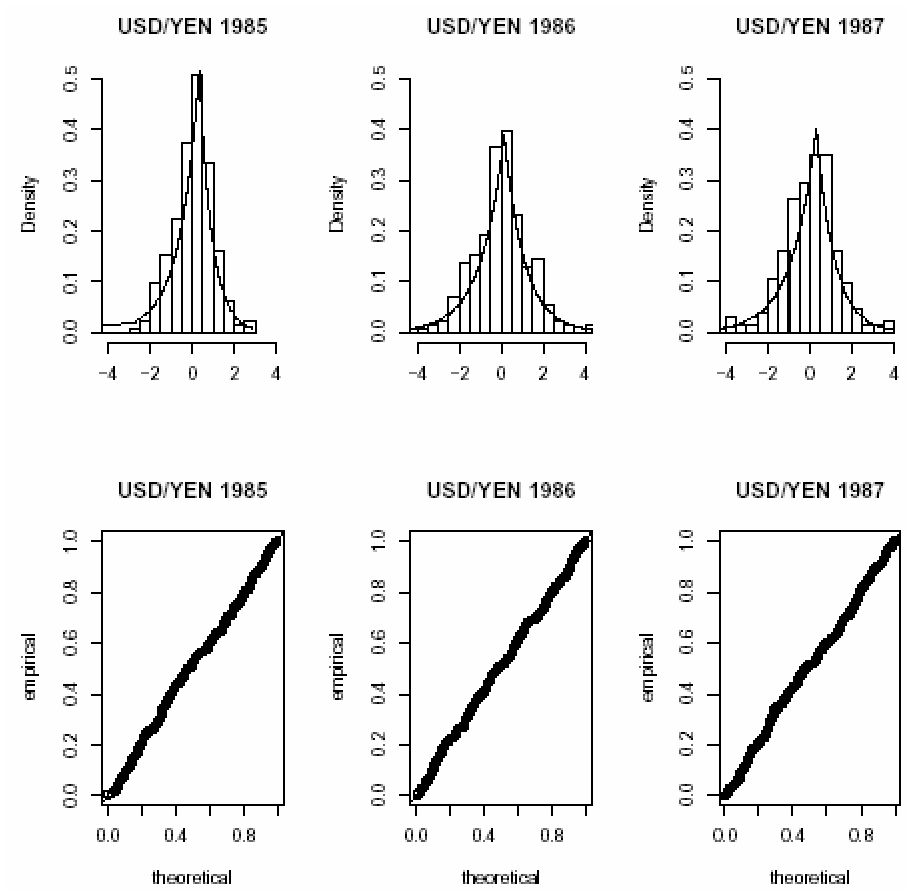

Figure 1. Histograms with fitted density and P-P plots for the USD/YEN exchange rate for 1985-1987.

■